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Technical Note

Skin friction and heat transfer in power-law fluid laminar boundary layer along a moving surface

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Abstract

Analytical and numerical solutions are presented for momentum and energy laminar boundary layer along a moving plate in power-law fluids utilizing a similarity transformation and shooting technique. The results indicate that for a given power-law exponent n $(0 < n \leqslant 1)$ or velocity ratio parameter ξ, the skin friction σ decreases with the increasing in ξ or n. The shear force decreases with the increasing in dimensionless tangential velocity t. When Prandtl number $N_{\mathrm{Pr}} = 1$, the dimensionless temperature $w(t)$ is a linear function of t, and the viscous boundary layer is similar to that of thermal boundary layer. In particular, $w(t) = t$ if $\xi = 0$, i.e., the velocity distribution in viscous boundary layer has the same pattern as the temperature distribution in the thermal boundary and $\delta = \delta_{\mathrm{T}}$. For $N_{\mathrm{Pr}} \geqslant 1$, the increase of viscous diffusion is larger than that of thermal diffusion with the increasing in N_{Pr}, and $\delta_{\mathrm{T}}(t) < \delta(t)$. The thermal diffusion ratio increases with the increasing in n $(0 < n \leqslant 1)$ and ξ.

Keywords: Boundary layer; Power-law fluid; Skin friction; Similarity solution; Shooting technique

1. Introduction

Fluid dynamicists have long known that the appearance of boundary layers was not restricted to the canonical problem of the motion of a body through a viscous fluid. Several other technologically important sources of boundary layer phenomenon are the flows behind expansion and shock waves traveling over smooth surfaces and the flow field above a moving conveyor belt [1,2,6–8,10,11].

The drag force due to "skin friction" is a fluid dynamic resistive force, which is a consequence of the fluid and the pressure distribution on the surface. Understanding the nature of this force by mathematical modeling with a view to predicting the drag forces and the associated behavior of fluid flow has been the focus of considerable research. A principal reason for the interest in analysis of boundary layer flows along solid surfaces is the possibility of applying the theory to the efficient design of supersonic and hypersonic flights. In addition, the mathematical model considered in present paper has significance in studying several problems of engineering, meteorology, and oceanography [3–6,8–11].

The purpose of this paper is to investigate the applicability of boundary layer theory for the flow of power-law fluids on a moving plate. A special emphasis is given to the formulation of boundary layer equations, which provide similarity solutions.

2. Laminar boundary layer equations

Consider a flat plate aligned with a uniform power-law flow at constant speed U_∞ and moving in the direction of the stream at constant speed U_{w}. In the absence of body force, external pressure gradients and viscous

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Nomenclature

c_p	specific heat
k	thermal conductivity
K, n	parameters in the power-law model, Eq. (4)
L	characteristic length
N_{Pr}	Prandtl number ($N_{Pr} = (c_p U_\infty \rho L)/k(N_{Re})^{2/(n+1)}$)
N_{Re}	Reynolds number ($N_{Re} = (\rho U^{2-n} L^n)/K$)
N_{Pe}	conventional Peclet number
T	temperature
T_w	surface temperature
T_∞	free stream temperature
U_∞	characteristic velocity
U	velocity component along x
V	velocity component along y
u, v	dimensionless velocity components defined by Eq. (7)
θ, ϕ, w	dimensionless temperature defined by Eq. (7)
X	distance along the surface from the leading edge, x dimensionless distance defined by (7)
Y	distance normal to the surface, y dimensionless distance defined by (7)
ψ	stream function, f dimensionless stream function defined by (7)
τ_{XY}	shear stress, $g(t)$ dimensionless shear stress
t	dimensionless tangential velocity ($t = f'(\eta)$)
σ	shear friction
δ	viscous boundary layer thickness
δ_T	thermal boundary layer thickness
ξ	velocity ratio parameter ($\xi = U_w/U_\infty$ and $0 \leqslant \xi < 1$)

dissipation, the laminar boundary layer equations expressing conservation of mass, momentum and energy can be written as follows (Fig. 1) [3,4,6,10,11]:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{1}$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \frac{1}{\rho}\frac{\partial \tau_{XY}}{\partial Y}, \tag{2}$$

$$U\frac{\partial T}{\partial X} + V\frac{\partial T}{\partial Y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial Y^2}, \tag{3}$$

where the X and Y axes are taken along and perpendicular to the plate, U and V are the velocity components parallel and normal to the plate, and

$$\tau_{xy} = K\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial U}{\partial Y} \tag{4}$$

is the shear stress with K as a positive constant. The case $n = 1$ corresponds to a Newtonian fluid and the case $0 < n < 1$ is "power-law" relation proposed as being descriptive of pseudo-plastic non-Newtonian fluids. The appropriate boundary conditions are

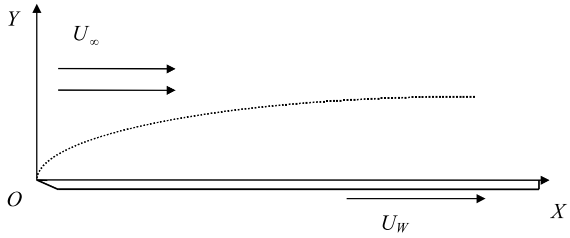

Fig. 1. Boundary layer structure on a moving flat plate.

$$U|_{y=0} = U_w, \quad V|_{y=0} = 0, \quad U|_{y=+\infty} = U_\infty, \tag{5}$$

$$T|_{y=0} = T_w, \quad T|_{y=+\infty} = T_\infty. \tag{6}$$

The following dimensionless variables are introduced:

$$x = \frac{X}{L}, \quad y = \left[\frac{\rho U_\infty^{2-n} L^n}{K}\right]^{1/(n+1)}\frac{Y}{L}, \quad u = \frac{U}{U_\infty},$$
$$v = \left[\frac{\rho U_\infty^{2-n} L^n}{K}\right]^{1/(n+1)}\frac{V}{U_\infty}, \quad \theta = \frac{T - T_w}{T_\infty - T_w},$$
$$\tau_{xy} = \left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y} = \left(\frac{K U^{3n}\rho^n}{L^n}\right)^{-1/(n+1)}\tau_{XY},$$
$$N_{Re} = \left[\frac{\rho U_\infty^{2-n} L^n}{K}\right], \quad N_{Pr} = \frac{c_p U_\infty \rho L}{k N_{Re}^{2/(n+1)}}. \tag{7}$$

The boundary layer equations then become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{8}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial \tau_{xy}}{\partial y}, \tag{9}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{N_{Pr}}\frac{\partial^2 \theta}{\partial y^2} \tag{10}$$

with the boundary conditions

$$u|_{y=0} = U_w/U_\infty, \quad v|_{y=0} = 0, \quad u|_{y=+\infty} = 1, \tag{11}$$

$$\theta|_{y=0} = 0, \quad \theta|_{y=+\infty} = 1. \tag{12}$$

Boundary layer equations (8)–(12) are super-non-linear and have a moving boundary. The problems are very complex to solve analytically and the numerical simulation is even difficult. So a similarity solution may be a considerable approach.

3. Converting into two-point boundary value problem

3.1. Stream function and similarity variable

The stream function $\psi(x, y)$, similarity variable η and dimensionless temperature function $\phi(\eta)$ are defined as

$$\psi = Ax^{\alpha} f(\eta), \quad \eta = Bx^{\beta} y, \quad \theta(x, y) = \phi(\eta), \tag{13}$$

where A, B, α and β are constants to be determined, and $f(\eta)$ denotes the dimensionless stream function. Thus, the u velocity components are

$$u = \frac{\partial \psi}{\partial y} = ABx^{\alpha+\beta} f'(\eta). \tag{14}$$

Choosing $\beta = -\alpha$ and $AB = 1$, then

$$v = -\frac{\partial \psi}{\partial x} = -A\alpha x^{\alpha-1}[f(\eta) - \eta f'(\eta)]. \tag{15}$$

Eq. (8) is satisfied automatically. Substituting u and v defined by (14) and (15) into (9)–(12) combining with

$$\alpha = \frac{1}{n+1} \quad \text{and} \quad B = \left(\frac{1}{(n+1)K}\right)^{1/(n+1)} \tag{16}$$

yields:

$$-f(\eta)f''(\eta) = (|f''(\eta)|^{n-1} f''(\eta))', \tag{17}$$

$$f(0) = 0, \quad f'(0) = \xi, \quad f'(\eta)\Big|_{\eta=+\infty} = 1, \tag{18}$$

$$\phi''(\eta) + N_{\mathrm{Pr}} f(\eta)\phi'(\eta) = 0, \tag{19}$$

$$\phi(0) = 0, \quad \phi(\eta)\Big|_{\eta=+\infty} = 1, \tag{20}$$

where $\xi = U_{\mathrm{w}}/U_{\infty}$ is the velocity ratio parameter ($n = 1$ and $\xi = 0$ corresponds to the classical Blasius problems). Unless otherwise indicate, in present paper, we always pay our attention to the case of $0 \leqslant \xi < 1$.

3.2. General Crocco variable transformation

We assume that the solution of Eqs. (17)–(20) possesses a positive second derivative $f''(\eta)$ in $(0, +\infty)$ (i.e., no boundary layer separation occurs) and $f''(+\infty) = 0$. Defining the general Crocco variable transformation as

$$g(t) = [f''(\eta)]^n, \quad w(t) = \phi(\eta), \quad t = f'(\eta), \quad t \in [\xi, 1), \ 0 \leqslant \xi < 1, \tag{21}$$

where t is the dimensionless tangential velocity, $g(t)$ is the dimensionless shear force, $w(t)$ is the dimensionless temperature. Substituting (21) into (17)–(20) and applying the chain rule yield the following singular non-linear two-point boundary value problems:

$$g''(t) = -tg^{-1/n}(t), \quad 0 \leqslant \xi < t < 1, \tag{22}$$

$$g'(\xi) = 0, \quad g(1) = 0, \tag{23}$$

$$w''(t)g(t) + (1 - N_{\mathrm{Pr}})w'(t)g'(t) = 0, \tag{24}$$

$$w(\xi) = 0, \quad w(1) = 1. \tag{25}$$

Eqs. (22) and (23) are obviously de-coupled and may be solved first, and the solutions then may be used to solve Eqs. (24) and (25).

In the two-point boundary value problem (22) and (23), the tangential velocity t is the independent variable, and the shear stress $g(t)$ is the dependent variable called as Crocco variable. Clearly, it may be seen from the derivation process that only the positive solutions of Eqs. (22) and (23) are physically significant.

4. Solving two-point boundary value problems

4.1. Solutions of Eqs. (22) and (23)

For the positive solutions of Eqs. (22) and (23), Callegari and Nachman [7] established the uniqueness and analyticity results for its special cases of $n = 1$ (Newtonian fluids) and showed that for each $0 \leqslant \xi < 1$, the problem has a unique positive solution which is analyticity about t in $[\xi, 1)$.

Recently, Zheng et al. [8–11] discussed some general cases of power-law fluid boundary layer equations for $0 < n \leqslant 1$ and some general non-linear boundary value problems corresponding to the surface moving in the direction or opposite to the direction of the stream. Sufficient conditions for existence, non-uniqueness, uniqueness and analytical positive solutions to the problem were obtained utilizing the perturbation and shooting techniques. They show that for each $0 \leqslant \xi < 1$, Eqs. (22) and (23) have a unique positive solution, the solution is analyticity about t in $[\xi, 1)$ and has a power series expansion

$$g(t) = \sum_{i=0}^{\infty} \frac{g^{(i)}(k)}{i!}(t - \xi)^i, \tag{26}$$

which converges at $t = 1$ to

$$g(1) = 0 = \sum_{i=0}^{\infty} \frac{g^{(i)}(\xi)}{i!}(1 - \xi)^i.$$

Here $g^{(i)}(\xi)$ can be established by the induction

$$g^{(m+3)}(\xi) = -g^{-1}\left[m!g^{1-(1/n)}\sum_{\lambda=p(m)} g^{-|\lambda|}\binom{1-\frac{1}{n}}{\lambda}\prod_{i=1}^{m}\left(\frac{g^{(i)}}{i!}\right)^{\lambda(i)} + \frac{1}{n}\sum_{i=0}^{m}\binom{m}{i}g^{(i+1)}g^{(m+2-i)} + \sum_{i=0}^{m}\binom{m}{i}g^{(i)}g^{(m+3-i)}\right], \tag{27}$$

where

$$g^{(i)} = \frac{\mathrm{d}^i g}{\mathrm{d}t^i}, \quad \binom{m}{i} = \frac{m!}{i!(m-i)!},$$

$p(m)$ is a partition of the integer m, and λ is a vector of the partition, whose first component $\lambda(1)$ is the number 1's in the partition and second component $\lambda(2)$ is the number 2's in the partition, etc.

$$|\lambda| = \sum_{i=1}^{m}\lambda(i),$$

$$\binom{p}{\lambda} = p(p-1)\cdots(p-|\lambda|+1)/(1)!(2)!\cdots\lambda(m)!$$

and the first sum is a sum over all partition m.

Eq. (27) indicates that each derivative of $g(t)$, of fourth or higher order, can be expressed in terms of those lower order, thus all derivatives of $g(t)$ depend only on the first three. Let $g(\xi) = \sigma$ (skin friction), then $g'(\xi) = 0$, $g''(\xi) = -\xi\sigma^{-1/n}$, $g'''(\xi) = -\sigma^{-1/n}$.

Eqs. (22) and (23) were solved for different values of n $(0 < n \leqslant 1)$ and ξ $(0 \leqslant \xi < 1)$ utilizing the shooting technique. The results are presented in Figs. 2–4. It may be seen that for each n $(0 < n \leqslant 1)$, the skin friction σ decreases with increasing in ξ, and this behavior is qualitatively true with n, i.e., for each ξ $(0 \leqslant \xi < 1)$ the skin friction σ decreases with increasing of n. For each ξ or n, the shear force $g(t)$ decreases with the increasing in $t \in [\xi, 1]$. The largest skin friction $g(\xi) = \sigma$ occurs at $t = \xi$ with the smallest shear force $g(1) = 0$ at $t = 1$.

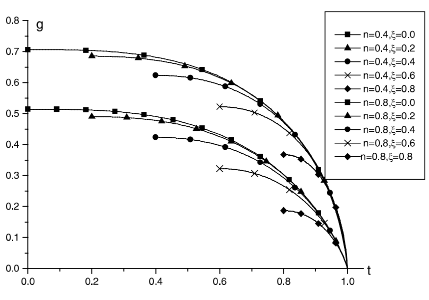

Fig. 2. Shear force distribution for: (i) $n = 0.4$, $\xi = 0.0$–0.8; (ii) $n = 0.8$, $\xi = 0.0$–0.8.

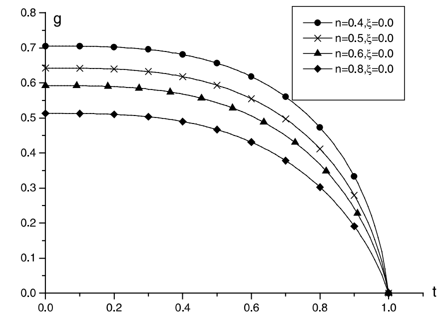

Fig. 3. Shear force distribution for $\xi = 0.0$, $n = 0.4$–0.8.

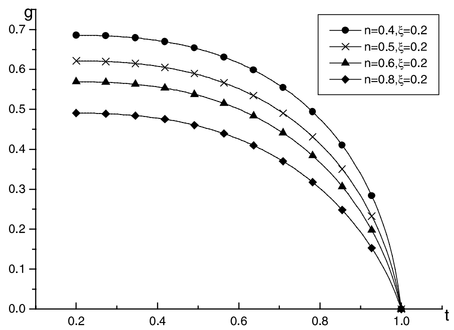

Fig. 4. Shear force distribution for $\xi = 0.2$, $n = 0.4$–0.8.

4.2. Solutions of Eqs. (24) and (25)

Eq. (24) is equivalent to

$$\frac{w''}{w'} = (N_{\mathrm{Pr}} - 1)\frac{g'}{g}. \tag{28}$$

Integrating Eq. (28) over $[\xi, t]$ with boundary condition (25) yields the analytical solution for the heat transfer, which can be represented as

$$w(t) = \frac{\int_{\xi}^{t} g^{N_{\mathrm{Pr}}-1}(s)\,\mathrm{d}s}{\int_{\xi}^{1} g^{N_{\mathrm{Pr}}-1}(s)\,\mathrm{d}s}. \tag{29}$$

When $N_{\mathrm{Pr}} = 1$, Eq. (29) gives $w(t) = (t-\xi)/(1-\xi)$. It indicates that the dimensionless temperature distribution is a linear function of dimensionless velocity t, which implies that the thermal boundary layer is similar to the viscous boundary layer. In particular, when $\xi = 0$, we obtain $w(t) = t$, which means that the temperature distribution in the thermal boundary layer is the same as the velocity distribution in viscous boundary layer and $\delta = \delta_{\mathrm{T}}$.

Eqs. (24) and (25) were solved numerically for $N_{\mathrm{Pr}} = 1, 2, 3, \ldots, 8$, $0 < n \leqslant 1$ and $0 \leqslant \xi < 1$ utilizing the shooting technique. The results are presented in Figs. 5–7, which illustrate the relations between momentum diffu-

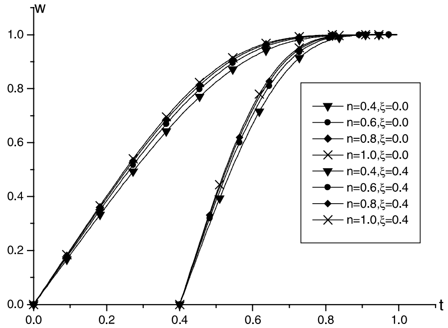

Fig. 5. Temperature for: (i) $N_{\mathrm{Pr}} = 8.0$, $\xi = 0.0$; (ii) $N_{\mathrm{Pr}} = 8.0$, $\xi = 0.4$.

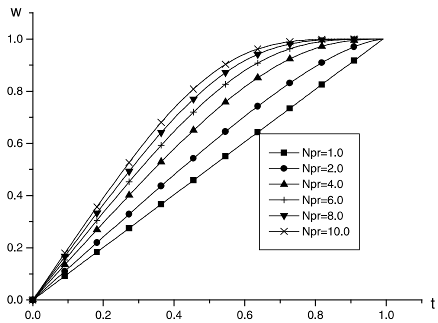

Fig. 6. Temperature distribution for $n = 0.4$, $\xi = 0.0$.

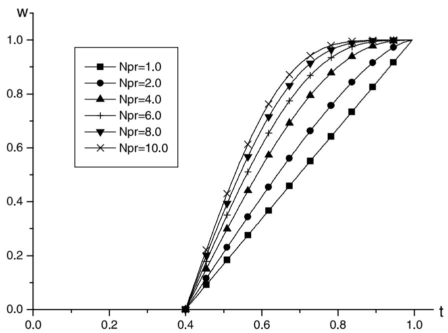

Fig. 7. Temperature distribution for $n = 0.4$, $\xi = 0.4$.

sion and thermal diffusion, as well as the effects of parameters n and ξ on the thermal diffusion.

Fig. 5 shows that for each N_{Pr} ($N_{\mathrm{Pr}} \geqslant 1$), the thermal diffusion ratio increases with the increasing in n ($0 < n \leqslant 1$), and this phenomenon is more obvious as N_{Pr} increasing. Figs. 6 and 7 show that for each ξ ($0 \leqslant \xi < 1$) and n ($0 < n \leqslant 1$), the thermal diffusion length scale can be significantly less than that of viscous diffusion with the increasing in N_{Pr}. Therefore the viscous diffusion rate exceeds the thermal diffusion rate. Obviously, the thermal boundary layer is thinner than the viscous boundary, i.e., $\delta_{\mathrm{T}}(t) < \delta(t)$.

5. Conclusions

Suitable similarity transformations were used to reduce the power-law fluid laminar boundary layer equations of momentum and energy to a class of singular non-linear boundary value problems. Analytical and numerical solutions were obtained.

The results showed that skin friction σ decreases with increasing in velocity ratio ξ, the shear force $g(t)$ decreases with increasing t in $[\xi, 1]$ and for a given ξ, a small n power-law fluid exerts a greater shear stress on the plate. When $N_{\mathrm{Pr}} = 1$, the thermal boundary layer has a similar form as the viscous boundary layer. In particular, when $\xi = 0$, the temperature distribution is the same as the velocity distribution and $\delta = \delta_{\mathrm{T}}$. With the increasing in N_{Pr} ($N_{\mathrm{Pr}} > 1$), the thermal diffusion length scale can be significantly less than that of viscous diffusion length scale. Therefore, $\delta_{\mathrm{T}}(t) < \delta(t)$. For a given N_{Pr} ($N_{\mathrm{Pr}} > 1$), the thermal diffusion increases with the increasing in n ($0 < n \leqslant 1$) and this phenomenon is more obvious as N_{Pr} increasing.

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References

[1] H. Mires, Boundary layer behind a shock or thin expansion wave moving into a stationary fluid, TH-3712, 1956, NACA.
[2] P.A. Thompson, in: Compressible – Fluid Dynamics, McGraw-Hill, New York, 1972, pp. 502–514.
[3] H. Schlichting, Boundary Layer Theory, McGraw-Hill, New York, 1979.
[4] A. Nachman, A. Callegari, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, SIAM J. Appl. Math. 38 (2) (1980) 275–281.
[5] J.J. Shu, G. Wilks, Heat transfer in the flow of a cold two-dimensional vertical liquid jet against a hot, horizontal plate, Int. J. Heat Mass Transfer 39 (16) (1996) 3367–3379.
[6] T.G. Hopwell, Momentum and heat transfer on a continuous moving surface in power law fluid, Int. J. Heat Mass Transfer 40 (8) (1997) 1853–1861.
[7] A.J. Callegari, A. Nachman, Some singular, non-linear differential equations arising in boundary layer theory, J. Math. Anal. Appl. 64 (1978) 96–105.

[8] L.C. Zheng, L.X. Ma, J.C. He, Bifurcation solutions to a boundary layer problem arising in the theory of power law fluids, Acta Math. Sci. 20 (1) (2000) 19–26.
[9] L.C. Zheng, X.Y. Deng, Singular nonlinear boundary value problem arising in the theory of viscous fluids, Acta Math. Sci. 20 (S) (2000) 577–582.
[10] L.C. Zheng, J.C. He, Existence and non-uniqueness of positive solutions to a non-linear boundary value problems in the theory of viscous fluids, Dyn. Syst. Appl. 8 (1999) 133–145.
[11] L.C. Zheng, X.Y. Deng, Y.M. Fan, Flat plate boundary layer problems with special suction/injection conditions in power law fluids, Acta Mech. Sci. 33 (5) (2001) 675–678.